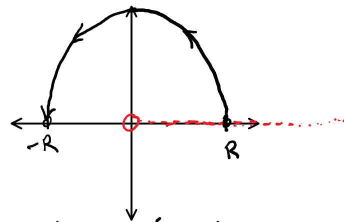
Some examples involving a branch of a multiple valued function:

(4) Integrate the branch of square root

$$f(z) = z^{1/2} = e^{\frac{1}{2}\log z} \quad (|z|>0,\ 0<\arg z<2\pi)$$

along the contour:

$$C: \quad z(t) = Re^{it} \quad (R>0,\ \underbrace{0\le t\le \pi})$$



The problem is that the integrand $\underline{f(z(t))z'(t)}$ is not defined when $t=0$. But the function is piecewise continuous on $[0,\pi]$:

$$f(z(t))\,z'(t) = e^{\frac{1}{2}\log Re^{it}}\, Rie^{it} = e^{\frac{1}{2}(\ln R + it)}\, Re^{it}$$
$$= \sqrt{R}\ e^{\frac{1}{2}it}\, Re^{it}$$
$$= R^{3/2}\ e^{3/2\, it} = R^{3/2}\left(\cos\tfrac{3}{2}t + i\sin\tfrac{3}{2}t\right)$$

The real/im parts of the function are continuous on $[0,\pi]$ and the limits approaching 0 from the right are as expected. So So the integrand is piecewise cont. on $[0,\pi]$ and the integral exists. We have

$$\int_C f(z)\,dz = R^{3/2}\int_0^{\pi} e^{3/2\, it}\,dt = R^{3/2}\left[\frac{2}{3i}e^{3/2\, it}\right]_0^{\pi}$$
$$= R^{3/2}\,\frac{2}{3i}\left(e^{3/2\,\pi i} - 1\right) = R^{3/2}\,\frac{2}{3i}\,(-i-1). \;//$$

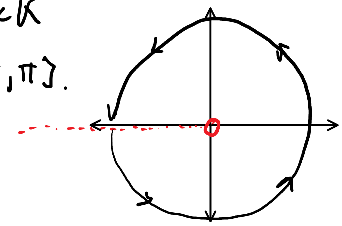
(5) Integrate the principal branch of z^{i-1}:

$$f(z) = z^{i-1} = e^{(i-1)\,\mathrm{Log}\, z}$$

along the contour

$$C: \quad z(t) = e^{it}, \quad -\pi \le t \le \pi.$$

The curve crosses the branch cut. We need to check if $f(z(t)) \cdot z'(t)$ piecewise continuous on $[-\pi, \pi]$.



We have

$$f(z(t))\, z'(t) = e^{(i-1)\,\mathrm{Log}\, e^{it}}\; i e^{it}$$

$$= e^{(i-1)(\ln 1 + it)}\; i e^{it} = e^{(i^2 - i)t} \cdot i e^{it} = i e^{-t}.$$

Hence,

$$\int_C f(z)\, dz = i \int_{-\pi}^{\pi} e^{-t} = i\left[-e^{-t}\right]_{-\pi}^{\pi}$$

$$= i\left[-e^{-\pi} + e^{\pi}\right]$$

$$= 2i \sinh \pi.$$

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Estimating Contour Integrals

Lemma (Triangle Ineq. for Integrals) Suppose $w : [a,b] \to \mathbb{C}$ is piecewise continuous. Then

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt.$$

Proof. First, assume $\int_a^b w(t)\, dt = 0$. Then the lemma holds since $|w(t)| \ge 0$ for all $t \in [a,b]$ and so its integral is also nonnegative. Otherwise, $\int_a^b w(t)\, dt \ne 0$ so we can use polar coordinates:

$$r_0 e^{i t_0} = \int_a^b w(t)\, dt.$$

Then

$$\left| \int_a^b w(t)\,dt \right| = \left| r_0 e^{it_0} \right|$$
$$= r_0$$
$$= \operatorname{Re} r_0$$
$$= \operatorname{Re}\left(r_0 e^{it_0} \cdot e^{-it_0}\right)$$
$$= \operatorname{Re}\left(\int_a^b e^{-it_0}\, w(t)\,dt \right)$$
$$= \int_a^b \operatorname{Re}\left(e^{-it_0}\, w(t)\right) dt$$

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$$\leq \int_a^b \left|e^{-it_0}\right| \left|w(t)\right| dt$$
$$= \int_a^b |w(t)|\,dt.$$

■

$\boxed{\text{Theorem (Triangle Ineq. for Contour Integrals)}}$ Suppose that C is a contour of length L and f is piecewise continuous on C. Then

finite?

$$\left| \int_C f(z)\,dz \right| \leq \max_{z \in C} |f(z)| \cdot L.$$

Proof. Suppose $z : [a,b] \to \mathbb{C}$ parameterizes C. By assumption $f(z(t))$ is piecewise continuous on $[a,b]$. Hence,

$$\max_{z \in C} |f(z)| = \max_{t \in [a,b]} |f(z(t))| \quad \text{is finite}$$

because $f(z(t))$ is piecewise cont. on a closed interval. Hence,

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t))\, z'(t)\,dt \right|$$
$$\leq \int_a^b \underbrace{|f(z(t))|}_{\leq \max_{z \in C} |f(z)|} |z'(t)|\,dt$$

Lemma

$$\leq \max_{z \in C} |f(z)| \cdot \int_a^b |z'(t)|\,dt$$
$$= \max_{z \in C} |f(z)| \cdot L.$$

■

$$\boxed{\text{Example}}$$

(1) Finding an upper bound for

$$\int_C \frac{z^2+1}{z^3+2}\,dz$$

Semicircle radius 2

where C is the semicircle $z(t) = 2e^{it}$, $0 \le t \le \pi$.

All we need to do is find $M \ge 0$ such that

$$\left| \frac{z^2+1}{z^3+2} \right| \le M \quad \text{for all } z \in C.$$

Suppose $z \in C$ so that $|z| = 2$. Then

$$|z^2+1| \le |z|^2 + 1 = 5.$$

Also,

$$|z^3+2| \ge \big||z|^3 - 2\big| = |2^3 - 2| = 6.$$

Together,

$$\left| \frac{z^2+1}{z^3+2} \right| \le \frac{5}{6} \quad \text{for all } z \in C.$$

Hence,

$$\left| \int_C \frac{z^2+1}{z^3+2}\,dz \right| \le \frac{5}{6} \cdot \overbrace{2\pi}^{L} \quad \text{by the Theorem.} \quad //$$

(2) Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^2+z}{z^4+2z^2+1}\,dz = 0$$

where C_R is the circle $z(t) = Re^{it}$, $0 \le t \le 2\pi$.

Note: the length of C_R is $2\pi R$. Let $z \in C_R$ so that $|z| = R$. Then

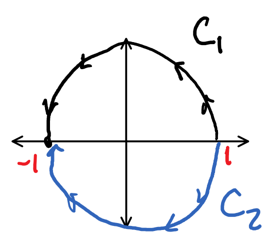
$$|z^2+z| \le |z|^2 + |z| = R^2 + R$$

and
$$|z^4+2z^2+1| = |(z^2+1)(z^2+1)|$$
$$= |z^2+1|^2 \qquad \left(\text{Assume } R>2\right)$$
$$\geq \big||z|^2-1\big|^2 = |R^2-1|^2$$
$$= (R^2-1)^2.$$

then
$$\left|\int_{C_R} \frac{z^2+z}{z^4+2z^2+1}\right| \leq 2\pi R \cdot \left(\frac{R^2+R}{(R^2-1)^2}\right) \xrightarrow{R\to\infty} 0. \quad //$$

Antiderivatives & Fundamental Theorem of Contour Integrals

Suppose C is a contour joining z_1 to z_2. In general, the value of the integral
$$\int_C f(z)\,dz$$
depends on C. For example, we have seen that
$$\int_{C_1} \frac{1}{z}\,dz = \pi i$$



while
$$\int_{C_2} \frac{1}{z}\,dz = -\pi i.$$

But we have also seen that
$$\int_C z\,dz = \frac{z_2^2 - z_1^2}{2}$$

difference between thes functions turns out to be that $f(z) = z$ has an antiderivative on $\mathbb{C}$, while $g(z) = \frac{1}{z}$ does not have an antiderivative on any domain containing C_1 and C_2. //

Definition (Antiderivative) Suppose that f is a continuous function on a domain D. An analytic function $F : D \to \mathbb{C}$ is called an *antiderivative* of f if $F'(z) = f(z)$ for all $z \in D$. //

Definition (Independence of Path) Let $f : D \to \mathbb{C}$ be a continuous function on a domain D and fix $z_1, z_2 \in D$. If

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

whenever C_1 and C_2 are contours in D joining z_1 to z_2, then the integrals of f from z_1 to z_2 are *independent of path* and we denote the unique value by

$$\int_{z_1}^{z_2} f(z)\, dz .$$

//

So, for instance we would write

$$\int_{z_1}^{z_2} z\, dz = \frac{z_2^2 - z^2}{2} .$$

Since we have already proved the integrals of z from z_1 to z_2 are independent of path.

Theorem (Fundamental Theorem of Contour Integrals) Suppose f is continuous on a domain D. The following are equivalent:

(1) f has an antiderivative $F : D \to \mathbb{C}$.

(2) For all $z_1, z_2 \in D$, the integrals of f from z_1 to z_2 are independent of path and the unique value is given by

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

(3) If C is any closed contour lying in D, then

$$\int_C f(z)\,dz = 0.$$

Proof. ((1) ⟹ (2)) Suppose f has an antiderivative $F : D \to \mathbb{C}$. Let $z_1, z_2 \in D$ and let C be any contour joining z_1 to z_2 and lying in D.

First assume C is a smooth arc parameterized by $z : [a,b] \to \mathbb{C}$.

Then $$\frac{d}{dt}\left(F(z(t))\right) \overset{\text{PSet 4 P2}}{=} F'(z(t))\,z'(t) = \underbrace{f(z(t))\,z'(t)}_{\text{the integrand}}$$

Hence (*) $$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$

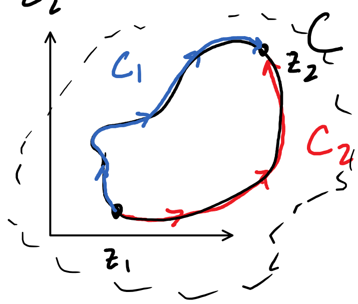
Now, assume C is a contour. Then $C = C_1 + \cdots + C_n$ where C_i is a smooth arc joining w_i and w_{i+1}. Then

where $w_{n+1} = z_2$ and $w_1 = z_1$

$$\int_C f(z)\,dz = \int_{\sum_{i=1}^n C_i} f(z)\,dz = \sum_{i=1}^n \int_{C_i} f(z)\,dz \overset{(*)}{=} \sum_{i=1}^n F(w_{i+1}) - F(w_i) = F(w_{n+1}) - F(w_1) = F(z_2) - F(z_1).$$

Since $F(z_2) - F(z_1)$ depends only on z_1 and z_2, we have proved the claim.

$((2) \Rightarrow (3))$ Assume (2) and let C be any closed contour lying in the domain. Choose any 2 distinct pts z_1 and z_2 on C. Let C_1 and C_2 be contours from z_1 to z_2 such that $C = C_1 - C_2$. Then



$$\int_C f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

by assumption

$$= \int_{z_1}^{z_2} f(z)\,dz - \int_{z_1}^{z_2} f(z)\,dz = 0.$$

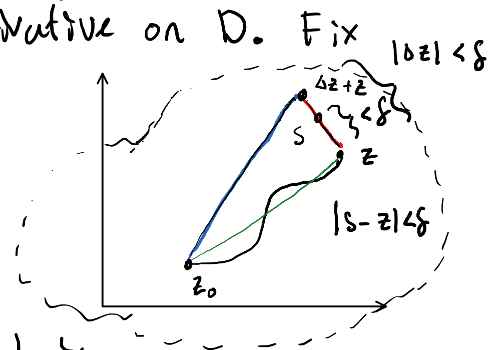
$((3) \Rightarrow (2))$ Assume (3) and let $z_1, z_2 \in D$. Suppose C_1 and C_2 are two contours in D joining z_1 to z_2. Then $C = C_1 - C_2$ is a closed contour. By assumption

by assumption

$$0 = \int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz.$$

So $\displaystyle\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ as claimed.

$((2) \Rightarrow (1))$ Assume (2) (and (3) since they are equivalent) - I need to show is that f has an antiderivative on D. Fix any point $z_0 \in D$ and define



$$F(z) = \int_{z_0}^{z} f(s)\,ds.$$

By (2), this function is well-defined. We need to show that $F'(z) = f(z)$, that is

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z).$$

Let $\varepsilon > 0$ and $z \in D$. Since f is continuous at z, so $\delta > 0$ such that

$$|s - z| < \delta \implies |f(s) - f(z)| < \varepsilon.$$

To compute the difference quotient, let Δz be a complex number close enough to z so that $z + \Delta z \in D$. Then

(both integrals taken over straight line paths)

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)\,ds - \int_{z_0}^{z} f(s)\,ds$$

integral over a closed path is 0

$$= \int_{z}^{z+\Delta z} f(s)\,ds.$$

Next,

PSet 4 P5

$$f(z) = \frac{f(z)\,\Delta z}{\Delta z} = \frac{1}{\Delta z} f(z) \int_z^{z+\Delta z} 1\,ds$$

$$= \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\,ds.$$

Now, assume Δz is is so close to z that $|\Delta z| < \delta$. It follows that $|s - z| < \delta$ for any point s on the line segment between z and $\Delta z + z$ (see picture). Hence, by continuity, $|f(s) - f(z)| < \varepsilon$. Using the preceding computations and the Triangle Ineq. for contour integrals, we obtain:

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{\left| \int_z^{z+\Delta z} f(s)\,ds - \int_z^{z+\Delta z} f(z)\,ds \right|}{|\Delta z|}$$

$$= \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} f(s) - f(z)\,ds \right|$$

length of line segment from z to $z+\Delta z$

$$\overset{\text{T.I.}}{\leq} \frac{1}{|\Delta z|}\, \varepsilon \cdot |\Delta z|$$

$$= \varepsilon.$$

We have shown that given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\Delta z| < \delta$ implies $\left| \dfrac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| < \varepsilon$.

That is $F'(z) = f(z)$ for all $z \in D$. ∎